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Asymptotic Expansion of Operator-Valued Laplace Transform

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1. INTRODUCTION

Let $f(t)$ be a locally integrable function on $[0, \infty)$, and let

$$\mathscr{L}_f(z) = \int_0^\infty f(t)e^{-zt}\,dt,$$

whenever the integral on the right converges. The well-known lemma of Watson [4] states that if $f(t)$ satisfies the following two conditions:

$$\text{(I)}\quad f(t) = \sum_{n=1}^{\infty} a_n t^{n/r-1}, \qquad |t| \leqslant c + \delta,$$

where r, c and δ are positive;

(II) There exist positive constants M_0 and b independent of t such that

$$|f(t)| < M_0 e^{bt}, \qquad t \geqslant c;$$

then

$$\mathscr{L}_f(z) \sim \sum_{n=1}^{\infty} a_n\,\Gamma(n/r) z^{-n/r},$$

as $z \to \infty$ in $|\arg z| \leqslant \pi/2 - \Delta$, $\Delta > 0$.

Recently Professor Luke asked the second author the following question. Consider the Laplace transform

$$\mathscr{L}_f(A) = \int_0^\infty f(t)e^{-tA}\,dt,$$

where the parameter A is a square matrix with positive eigenvalues. Is there an analog of Watson's lemma for this matrix-valued Laplace transform? The present paper is intended to answer this question affirmatively, when A is a normal matrix. We in fact prove a rather general result for an operator-valued Laplace transform, from which the matrix case is shown to follow.

2. Preliminaries

Let X be a Banach space over the complex field $\mathbb{C}$ and Y be a dense subspace of X. Let A be a closed linear operator from Y into X. The set $\rho(A)$ of complex numbers λ for which $\lambda I - A$ has a bounded inverse $R_\lambda(A) = (\lambda I - A)^{-1}$ on X, I being the identity operator, is called the *resolvent set* of A. The operator $R_\lambda(A)$ is called the *resolvent* of A. The *spectrum* of A, denoted by $\sigma(A)$, is the complement of $\rho(A)$. The number $r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}$ is called the *spectral radius* of A.

Lemma 1.

(a) *$\sigma(A)$ is a closed set.*

(b) *If A has a bounded inverse A^{-1} on X, $\lambda \neq 0$ and $\lambda \in \sigma(A)$, then $\lambda^{-1} \in \sigma(A^{-1})$.*

Proof. (a) is well known: see, for example, [5, p. 211]. To prove (b), we suppose that $\lambda^{-1} \in \rho(A^{-1})$. Let $B = (-\lambda^{-1}A^{-1}) R_{\lambda^{-1}}(A^{-1})$ and $C = R_{\lambda^{-1}}(A^{-1})(-\lambda^{-1}A^{-1})$. Note that B and C are bounded operators on X, and that

$$C(\lambda I - A)\, y = y \quad (y \in Y), \qquad (\lambda I - A)\, Bx = x \quad (x \in X).$$

Hence, $B = C$ and $\lambda I - A$ has a bounded inverse, i.e., $\lambda \in \rho(A)$, which is a contradiction.

Throughout this section we shall assume that A satisfies the following conditions:

(C_1) *There exists a positive Δ such that*

$$\sigma(A) \subseteq \{\lambda \in \mathbb{C}: \lambda \neq 0 \text{ and } |\arg \lambda| \leqslant \pi/2 - \Delta\}.$$

(C_2) *Let* $\omega(A) = \inf\{\mathrm{Re}\, \lambda : \lambda \in \sigma(A)\}$. *There exist* $M > 0$ *and* $0 < \omega_1 \leqslant \omega(A)$ *such that for any positive integer* n,

$$\| R_\lambda(A)^n \| \leqslant M/(\omega_1 - \lambda)^n$$

for all real $\lambda < \omega_1$.

Note that $\omega(A) > 0$, by Lemma 1(a).

Let $K = 1 + M(1 + \omega_1^{-1})$. Then for $\lambda \leqslant 0$,

$$\| R_\lambda(A) \| \leqslant K/(1 + |\lambda|).$$

Let Γ be the contour consisting of the two half-lines

$$\arg(\lambda - 1/4K) = \pm(\pi - \arcsin(1/2K)).$$

For any $0 < \alpha < \infty$, we define [3, p. 111]

$$A^{-\alpha} = \frac{1}{2\pi i} \int_\Gamma \lambda^{-\alpha} R_\lambda(A)\, d\lambda. \tag{1}$$

These are bounded linear operators (the integrals converge in norm), they form a semigroup, and for every positive integer n, $A^{-n} = (A^{-1})^n$.

Since A satisfies conditions C_1 and C_2, it follows from [2, p. 95] that there exists a strongly continuous semigroup, $\{e^{-tA}: 0 \leqslant t < \infty\}$, of bounded linear operators on X such that

$$\| e^{-tA} \| \leqslant M e^{-t\omega_1}, \qquad \text{for all} \quad 0 < t < \infty; \tag{2}$$

see [2, p. 99].

The following result is given in [3, p. 122]:

LEMMA 2. *For any* $0 < \alpha < \infty$,

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-tA}\, dt.$$

3. MAIN RESULTS

Let $\{A_\alpha\}$ be a net of closed linear operators, each of which satisfies conditions C_1 and C_2. Let $F(A_\alpha)$, $\varphi_1(A_\alpha), \ldots, \varphi_n(A_\alpha), \ldots$ be bounded linear operators depending on A_α. We say that $\{\varphi_n(A_\alpha)\}$ is an *asymptotic sequence* if for all $n \geqslant 1$

$$\| \varphi_{n+1}(A_\alpha) \| = o(\| \varphi_n(A_\alpha) \|), \qquad \text{as} \quad \| A_\alpha^{-1} \| \to 0.$$

The formal series

$$\sum_{n=1}^\infty a_n \varphi_n(A_\alpha),$$

is said to be an *asymptotic expansion* of $F(A_\alpha)$ if, for every value of $N \geqslant 1$,

$$\left\| F(A_\alpha) - \sum_{n=1}^N a_n \varphi_n(A_\alpha) \right\| = o(\| \varphi_N(A_\alpha) \|), \qquad \text{as} \quad \| A_\alpha^{-1} \| \to 0.$$

In this case we write

$$F(A_\alpha) \sim \sum_{n=1}^\infty a_n \varphi_n(A_\alpha), \qquad \text{as} \quad \| A_\alpha^{-1} \| \to 0.$$

Let r be a positive constant. By the moment inequality [3, p. 115],

$$\| A_\alpha^{-(n+1)/r} \| \leqslant \| A_\alpha^{-n/r} \| \cdot \| A_\alpha^{-1/r} \| \leqslant C_r \| A_\alpha^{-n/r} \| \cdot \| A_\alpha^{-1} \|^{1/r},$$

where C_r is a constant depending only on r. Hence

$$\varphi_n(A_\alpha) = A_\alpha^{-n/r}, \qquad n = 1, 2, \ldots,$$

forms an asymptotic sequence.

THEOREM 1. *Let $\{A_\alpha\}$ be a net of closed linear operators, each satisfying conditions C_1 and C_2 with the same Δ and M and such that there is some positive η with $\omega_1(A_\alpha) \geqslant \eta\omega(A_\alpha)$ for each A_α. If $f(t)$ is a function satisfying the conditions* (I) *and* (II) *of Watson's lemma, then the bounded linear operator $\mathscr{L}_f(A_\alpha)$ has the asymptotic expansion*

$$\mathscr{L}_f(A_\alpha) \sim \sum_{n=1}^{\infty} a_n \Gamma(n/r) A_\alpha^{-n/r}, \qquad \text{as} \quad \| A_\alpha^{-1} \| \to 0.$$

Proof. For convenience we let

$$\omega_{1,\alpha} = \omega_1(A_\alpha) \qquad \text{and} \qquad \omega_\alpha = \omega(A_\alpha).$$

By hypothesis,

$$\begin{aligned} \omega_{1,\alpha}^{-1} &\leqslant \eta^{-1} \sup\{(\operatorname{Re} \lambda)^{-1} : \lambda \in \sigma(A_\alpha)\} \\ &\leqslant \eta^{-1} \sup\{(|\lambda| \sin \Delta)^{-1} : \lambda \in \sigma(A_\alpha)\}. \end{aligned}$$

Hence it follows that

$$\begin{aligned} \omega_{1,\alpha}^{-1} &\leqslant \eta^{-1} \sup\{|\mu| (\sin \Delta)^{-1} : \mu \in \sigma(A_\alpha^{-1})\} \\ &= \eta^{-1} r(A_\alpha^{-1}) (\sin \Delta)^{-1} \leqslant \| A_\alpha^{-1} \| (\eta \sin \Delta)^{-1} \end{aligned} \tag{3}$$

in view of Lemma 1(*b*).

Now, fix an integer $N \geqslant 2$. Clearly, there exists a constant C such that for all $t \geqslant 0$, whether $t \leqslant c$ or $t > c$,

$$\left| f(t) - \sum_{n=1}^{N-1} a_n t^{n/r-1} \right| \leqslant C t^{N/r-1} e^{bt}. \tag{4}$$

By Lemma 2, we may write

$$\mathscr{L}_f(A_\alpha) - \sum_{n=1}^{N-1} a_n \Gamma(n/r) A_\alpha{}^{n/r} = \int_0^\infty \left[f(t) - \sum_{n=1}^{N-1} a_n t^{n/r-1} \right] e^{-tA_\alpha} \, dt.$$

If $\| A_\alpha^{-1} \| \leqslant \eta \sin \Delta/2b$, then $\omega_{1,\alpha} \geqslant 2b$ by (3) and hence

$$\left\| \mathscr{L}_f(A_\alpha) - \sum_{n=1}^{N-1} a_n \Gamma(n/r) A_\alpha^{-n/r} \right\| \leqslant CM \int_0^\infty t^{N/r-1} e^{bt} e^{-\omega_{1,\alpha} t} \, dt,$$

by virtue of (2) and (4). A simple calculation then gives

$$\left\| \mathscr{L}_f(A_\alpha) - \sum_{n=1}^{N-1} a_n \Gamma(n/r) A_\alpha^{-n/r} \right\| \leqslant K_1 \omega_{1,\alpha}^{-N/r} \leqslant K_2 \| A_\alpha^{-1} \|^{N/r}, \qquad (5)$$

where K_1 and K_2 are positive constants independent of A_α. The last inequality follows from (3). Applying the moment inequality [3, p. 115] twice, we have

$$\| A_\alpha^{-1} \|^{N/r} \leqslant C_1 \| A_\alpha^{-N/r} \| \leqslant C_1 \| A_\alpha^{-(N-1)/r} \| \cdot \| A_\alpha^{-1/r} \|$$
$$\leqslant C_2 \| A_\alpha^{-(N-1)/r} \| \cdot \| A_\alpha^{-1} \|^{1/r},$$

where C_1 and C_2 depend only on N and r. Therefore, (5) implies

$$\left\| \mathscr{L}_f(A_\alpha) - \sum_{n=1}^{N-1} a_n \Gamma(n/r) A_\alpha^{-n/r} \right\| = o(\| A_\alpha^{-(N-1)/r} \|) \quad \text{as} \quad \| A_\alpha^{-1} \| \to 0, \qquad (6)$$

thus proving the theorem.

COROLLARY. *Let $\{A_\alpha\}$ be a net of bounded linear operators on a Hilbert space $\mathscr{H}$. If each A_α is normal and satisfies condition C_1 with the same Δ, and if $f(t)$ satisfies conditions* (I) *and* (II) *of Watson's lemma, then*

$$\mathscr{L}_f(A_\alpha) \sim \sum_{n=1}^{\infty} a_n \Gamma(n/r) A_\alpha^{-n/r}, \qquad \textit{as} \quad \| A_\alpha^{-1} \| \to 0.$$

Proof. It suffices to show that each A_α satisfies condition C_2 with $M = 1$ and $\omega_1(A_\alpha) = \omega(A_\alpha)$. This follows immediately from the fact that for all real $\lambda < \omega(A_\alpha)$, we have

$$\| R_\lambda(A_\alpha)^n \| = \sup_{\mu \in \sigma(A_\alpha)} |(\lambda - \mu)^{-n}| \leqslant (\omega - \lambda)^{-n},$$

since A_α is normal [1, p. 879].

4. Remarks

The above corollary in particular covers the case when $\{A_\alpha\}$ is a net of $n \times n$ normal matrices. The spectrum $\sigma(A_\alpha)$ in this case is precisely the set of eigenvalues of A_α.

If the elements of A are denoted by a_{ij} then the operator norm

$$\| A \| = \sup_{\|x\|=1} \| Ax \|, \qquad x \in \mathbb{C}^n,$$

used above can be replaced by any one of the following

$$\| A \| = \sum_{i,j=1}^{n} | a_{ij} |, \quad \| A \| = \left(\sum_{i,j=1}^{n} | a_{ij} |^2 \right)^{1/2}, \quad \| A \| = \max_{1 \leqslant i \leqslant n} \sum_{j=1}^{n} | a_{ij} |,$$

since these norms are all equivalent. Furthermore, if A is a normal matrix then the fractional powers of A can be expressed in a simpler form. Let $\lambda_1, ..., \lambda_n$ be the eigenvalues of A and $D = \operatorname{diag}[\lambda_1, ..., \lambda_n]$. Since A is normal, there exists a unitary matrix U such that $\mathsf{U}^{-1} A \mathsf{U} = D$. By (1),

$$A^{-\alpha} = \frac{1}{2\pi i} \int_\Gamma \lambda^{-\alpha} R_\lambda(A)\, d\lambda$$

$$= \mathsf{U} \left(\frac{1}{2\pi i} \int_\Gamma \lambda^{-\alpha} R_\lambda(D)\, d\lambda \right) \mathsf{U}^{-1}, \qquad (0 < \alpha < \infty).$$

Since

$$\frac{1}{2\pi i} \int_\Gamma \frac{\lambda^{-\alpha}}{\lambda - \lambda_i}\, d\lambda = \lambda_i^{-\alpha},$$

we conclude that

$$A^{-\alpha} = \mathsf{U} D^{-\alpha} \mathsf{U}^{-1},$$

where $D^{-\alpha} = \operatorname{diag}[\lambda_1^{-\alpha}, ..., \lambda_n^{-\alpha}]$.

Finally we remark that in view of the conditions of Watson's lemma, it is tempting to conjecture that the result (6) can be improved to read

$$\left\| \mathscr{L}_f(A_\alpha) - \sum_{n=1}^{N-1} a_n \Gamma(n/r) A_\alpha^{-n/r} \right\| = o(\| A_\alpha \|^{-(N-1)/r}), \tag{7}$$

as $\| A_\alpha \| \to +\infty$. However, this conjecture is false even for diagonal matrices. To see this, we let $f(t) = 1 + t$ and $A_\alpha = \operatorname{diag}[1, \alpha, ..., \alpha^{n-1}]$, where α is a positive parameter tending to infinity. Clearly $\| A_\alpha \| = \alpha^{n-1} \to +\infty$ and

$$\| \mathscr{L}_f(A_\alpha) - A_\alpha^{-1} \| = \| A_\alpha^{-2} \| = 1.$$

Hence (7) is not satisfied.

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